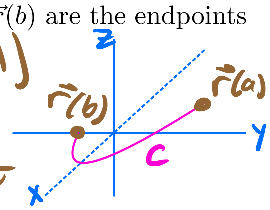
MATH 112A Review: Line Integrals, Surface Integrals, Parametrization of Curves

Facts to Know:

(Line integral of a scalar field) Let $f : \mathbb{R}^n \to \mathbb{R}$ be a scalar field and let $\vec{r}(t)$ be a bijective parametrization of a curve C in $\mathbb{R}^n$ with parameter $t \in [a, b]$ such that $\vec{r}(a)$ and $\vec{r}(b)$ are the endpoints of C. Then the line integral along C is

$\vec{r}(t) = (x(t), y(t))$



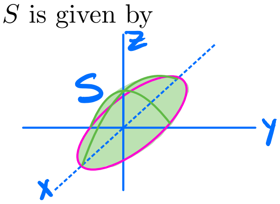
$$\int_C f(x_1, \ldots, x_n) ds = \int_a^b f(\vec{r}(t)) \|\vec{r}\,'(t)\| dt$$

(Line integral of a vector field) Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a vector field and let $\vec{r}(t)$ be a bijective parametrization of a curve C in $\mathbb{R}^n$ with parameter $t \in [a, b]$ such that $\vec{r}(a)$ and $\vec{r}(b)$ are the endpoints of C. Then the line integral along C is in the direction of $\vec{r}$ is

$$\int_C F(\vec{r}) \cdot d\vec{r} = \int_a^b F(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$$

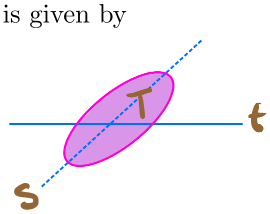
(Surface integral of a scalar field) Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a scalar field and let $\vec{r}(s, t)$ be a parametrization of a surface S in $\mathbb{R}^3$ with (s, t) vary in some region T in the plain. Then, the surface integral over S is given by

$\vec{r}(s,t) = (x(s,t), y(s,t), z(s,t))$



$$\iint_S f(x, y, z) dS = \iint_T f(\vec{r}(s,t)) \left\| \frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial t} \right\| ds\,dt$$

(Surface integral of a vector field) Let $F : \mathbb{R}^3 \to \mathbb{R}$ be a scalar field and let $\vec{r}(s, t)$ be a parametrization of a surface S in $\mathbb{R}^3$ with (s, t) vary in some region T in the plain. Then, the surface integral over S is given by



$$\iint_S F \cdot d\vec{S} = \iint_T F(\vec{r}(s,t)) \cdot \left(\frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial t} \right) ds\,dt$$

Examples:

1. Let $F(x, y) = (P(x, y), Q(x, y))$ and let $\vec{r}(t) = (x(t), y(t))$ be the parametrization of C. What is the line integral?

$t \in [a, b]$

$$\int_C F(\vec{r}) \cdot d\vec{r} = \int_a^b F(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$$

$$= \int_a^b \big(P(x(t), y(t)), Q(x(t), y(t))\big) \cdot \big(x'(t), y'(t)\big) dt$$

$$= \int_a^b P(x(t), y(t))\, x'(t)\,dt + \int_a^b Q(x(t), y(t))\, y'(t)\,dt$$

2. Let $F(x, y) = (y, -x)$ and consider the parametrization $\vec{r}(t) = (\cos t, \sin t)$ for $t \in [0, 2\pi]$ of the unit circle C with counterclockwise orientation. Compute

$$\int_C F(\vec{r}) \cdot \vec{r}\,'(t)dt \quad = (*)$$

$$\vec{r}\,'(t) = (-\sin t, \cos t)$$

$$F(\vec{r}(t)) = (\sin t, -\cos t)$$

$$F(\vec{r}(t)) \cdot \vec{r}\,'(t) = -\sin^2 t - \cos^2 t$$

$$= -1$$

$$(*) = \int_0^{2\pi} -1\, dt = -2\pi$$